Exchange economy with quasilinear utility functions

In an economy there are two consumers, A and B, who consume two goods, good x and good y. The preferences of these consumers are given by the following utility functions:

$$U_A = x_A + 4 \cdot \ln(y_A)$$

$$U_B = x_B + 6 \cdot \ln(y_B)$$

Consumer A has an initial endowment $(\bar{x}_A = 8, \bar{y}_A = 8)$, while consumer B has an endowment $(\bar{x}_B = 2, \bar{y}_B = 12)$, that is, 2 units of good x and 12 units of good y. Since these consumers spend all of their income and take prices as given, answer the following:

1. What relative price equilibrates the market? What are the consumption bundles of both consumers after trade?
2. Suppose now that the government decides to "redistribute" wealth by taking 2 units of good x and 4 units of good y from consumer B and transferring them to consumer A. Does the equilibrium change?
3. What conditions must hold to ensure that we are in a market equilibrium (Walrasian equilibrium)?
4. Find the contract curve and the utility possibility frontier (UPF)
5. Verify that the equilibrium found in the first part is efficient, using your result from part 4.

1. Let prices be (p_x, p_y). Consumer A solves

$$\max_{x_A, y_A} \; x_A + 4\ln(y_A)$$

subject to

$$p_x x_A + p_y y_A = 8p_x + 8p_y$$

$$x_A \geq 0, \qquad y_A > 0$$

Since $\ln(y_A)$ is only defined for $y_A > 0$, there cannot be a corner solution with $y_A = 0$. The only possible corner is $x_A = 0$.

For an interior solution, the Lagrangian is

$$\mathcal{L}_A = x_A + 4\ln(y_A) + \lambda_A \left(8p_x + 8p_y - p_x x_A - p_y y_A\right)$$

The first-order conditions are

$$\frac{\partial \mathcal{L}_A}{\partial x_A} = 1 - \lambda_A p_x = 0$$

$$\frac{\partial \mathcal{L}_A}{\partial y_A} = \frac{4}{y_A} - \lambda_A p_y = 0$$

$$\frac{\partial \mathcal{L}_A}{\partial \lambda_A} = 8p_x + 8p_y - p_x x_A - p_y y_A = 0$$

From the first condition,

$$\lambda_A = \frac{1}{p_x}$$

Substituting into the second condition,

$$\frac{4}{y_A} = \frac{p_y}{p_x}$$

$$y_A = 4\frac{p_x}{p_y}$$

Using the budget constraint,

$$x_A = \frac{8p_x + 8p_y - p_y y_A}{p_x}$$

$$x_A = \frac{8p_x + 8p_y - p_y\left(4\frac{p_x}{p_y}\right)}{p_x}$$

$$x_A = \frac{4p_x + 8p_y}{p_x} = 4 + 8\frac{p_y}{p_x}$$

Since prices are positive, this implies

$$x_A > 0$$

so consumer A indeed has an interior solution.

Now consumer B solves

$$\max_{x_B, y_B} \; x_B + 6\ln(y_B)$$

subject to

$$p_x x_B + p_y y_B = 2p_x + 12p_y$$

$$x_B \geq 0, \qquad y_B > 0$$

For an interior solution, the Lagrangian is

$$\mathcal{L}_B = x_B + 6\ln(y_B) + \lambda_B \left(2p_x + 12p_y - p_x x_B - p_y y_B\right)$$

The first-order conditions are

$$\frac{\partial \mathcal{L}_B}{\partial x_B} = 1 - \lambda_B p_x = 0$$

$$\frac{\partial \mathcal{L}_B}{\partial y_B} = \frac{6}{y_B} - \lambda_B p_y = 0$$

$$\frac{\partial \mathcal{L}_B}{\partial \lambda_B} = 2p_x + 12p_y - p_x x_B - p_y y_B = 0$$

From the first condition,

$$\lambda_B = \frac{1}{p_x}$$

Substituting into the second condition,

$$\frac{6}{y_B} = \frac{p_y}{p_x}$$

$$y_B = 6\frac{p_x}{p_y}$$

Using the budget constraint,

$$x_B = \frac{2p_x + 12p_y - p_y y_B}{p_x}$$

$$x_B = \frac{2p_x + 12p_y - p_y\left(6\frac{p_x}{p_y}\right)}{p_x}$$

$$x_B = \frac{-4p_x + 12p_y}{p_x} = -4 + 12\frac{p_y}{p_x}$$

Thus, consumer B has an interior solution only if

$$-4 + 12\frac{p_y}{p_x} > 0$$

$$12\frac{p_y}{p_x} > 4$$

$$\frac{p_y}{p_x} > \frac{1}{3}$$

$$\frac{p_x}{p_y} < 3$$

Now impose market clearing in good y:

$$y_A + y_B = 20$$

$$4\frac{p_x}{p_y} + 6\frac{p_x}{p_y} = 20$$

$$10\frac{p_x}{p_y} = 20$$

$$\frac{p_x}{p_y} = 2$$

Since $2 < 3$, consumer B also has an interior solution at the equilibrium price.
Therefore,

$$y_A = 4 \cdot 2 = 8 \qquad y_B = 6 \cdot 2 = 12$$

Using the budget constraints,

$$x_A = \frac{8p_x + 8p_y - 8p_y}{p_x} = 8$$

$$x_B = \frac{2p_x + 12p_y - 12p_y}{p_x} = 2$$

Hence, the equilibrium allocation is

$$(x_A, y_A) = (8, 8) \qquad (x_B, y_B) = (2, 12)$$

Thus, the market-clearing relative price is $\frac{p_x}{p_y} = 2$, and after checking the nonnegativity constraints we confirm that the equilibrium is interior

2. After the redistribution, the new initial endowments are

$$(\bar{x}_A, \bar{y}_A) = (10, 12) \qquad (\bar{x}_B, \bar{y}_B) = (0, 8)$$

Total endowments in the economy do not change, so

$$\bar{x} = 10 \qquad \bar{y} = 20$$

Let prices be (p_x, p_y). Consumer A solves

$$\max_{x_A, y_A} \; x_A + 4\ln(y_A)$$

subject to

$$p_x x_A + p_y y_A = 10p_x + 12p_y$$

$$x_A \geq 0 \qquad y_A > 0$$

The Lagrangian is

$$\mathcal{L}_A = x_A + 4\ln(y_A) + \lambda_A \left(10p_x + 12p_y - p_x x_A - p_y y_A\right)$$

The first-order conditions are

$$\frac{\partial \mathcal{L}_A}{\partial x_A} = 1 - \lambda_A p_x = 0$$

$$\frac{\partial \mathcal{L}_A}{\partial y_A} = \frac{4}{y_A} - \lambda_A p_y = 0$$

$$\frac{\partial \mathcal{L}_A}{\partial \lambda_A} = 10p_x + 12p_y - p_x x_A - p_y y_A = 0$$

From the first two conditions,

$$\lambda_A = \frac{1}{p_x}$$

$$\frac{4}{y_A} = \frac{p_y}{p_x}$$

$$y_A = 4\frac{p_x}{p_y}$$

Now consider consumer B. She solves

$$\max_{x_B, y_B} \; x_B + 6\ln(y_B)$$

subject to

$$p_x x_B + p_y y_B = 8p_y$$

$$x_B \geq 0 \qquad y_B > 0$$

If consumer B had an interior solution, the same Lagrangian method would give

$$y_B = 6\frac{p_x}{p_y}$$

Using the budget constraint,

$$x_B = \frac{8p_y - p_y y_B}{p_x} = 8\frac{p_y}{p_x} - 6$$

Thus, an interior solution requires

$$8\frac{p_y}{p_x} - 6 > 0$$

$$\frac{p_y}{p_x} > \frac{3}{4}$$

$$\frac{p_x}{p_y} < \frac{4}{3}$$

We now check whether this can hold in equilibrium. If both consumers were interior, market clearing in good y would require

$$y_A + y_B = 20$$

$$4\frac{p_x}{p_y} + 6\frac{p_x}{p_y} = 20$$

$$10\frac{p_x}{p_y} = 20$$

$$\frac{p_x}{p_y} = 2$$

But this contradicts the condition

$$\frac{p_x}{p_y} < \frac{4}{3}$$

Therefore, consumer B cannot be interior in equilibrium, so her solution must be a corner Since consumer B has only 8 units of income measured in good y, the corner is

$$x_B = 0 \qquad y_B = 8$$

Now impose market clearing in good y

$$y_A + y_B = 20$$

$$4\frac{p_x}{p_y} + 8 = 20$$

$$4\frac{p_x}{p_y} = 12$$

$$\frac{p_x}{p_y} = 3$$

Using this price ratio,

$$y_A = 4 \cdot 3 = 12$$

and since consumer B is at the corner,

$$(x_B, y_B) = (0, 8)$$

Using consumer A's budget constraint,

$$p_x x_A + p_y(12) = 10p_x + 12p_y$$

$$x_A = 10$$

Hence,

$$(x_A, y_A) = (10, 12) \qquad (x_B, y_B) = (0, 8)$$

Yes, the equilibrium changes. The new market-clearing relative price is $\frac{p_x}{p_y} = 3$, and the new equilibrium allocation is $(10, 12)$ for consumer A and $(0, 8)$ for consumer B

The relative price does not generally remain unchanged after redistribution. With quasilinear preferences, redistribution does not affect the demand for good y only as long as both consumers remain at an interior solution. In this case, after the transfer consumer B has too little wealth to afford the interior demand for y, so B is pushed to the corner $(x_B, y_B) = (0, 8)$. As a result, aggregate demand changes and the market-clearing relative price rises from $\frac{p_x}{p_y} = 2$ to $\frac{p_x}{p_y} = 3$

3. A Walrasian equilibrium in this exchange economy is a price vector (p_x, p_y) with $p_x > 0$ and $p_y > 0$, together with an allocation

$$(x_A, y_A), (x_B, y_B)$$

such that the following conditions hold

(a) **Utility maximization**

Given prices, each consumer chooses the most preferred affordable bundle

For consumer A,

$$(x_A, y_A) \in \arg \max_{x_A \geq 0,\ y_A > 0} \{x_A + 4\ln(y_A) \ :\ p_x x_A + p_y y_A \leq p_x \bar{x}_A + p_y \bar{y}_A\}$$

For consumer B,

$$(x_B, y_B) \in \arg \max_{x_B \geq 0,\ y_B > 0} \{x_B + 6\ln(y_B) \ :\ p_x x_B + p_y y_B \leq p_x \bar{x}_B + p_y \bar{y}_B\}$$

Since both utility functions are increasing in x and in y, the budget constraints hold with equality at equilibrium

$$p_x x_A + p_y y_A = p_x \bar{x}_A + p_y \bar{y}_A$$

$$p_x x_B + p_y y_B = p_x \bar{x}_B + p_y \bar{y}_B$$

(b) **Feasibility and market clearing**

Total consumption must equal total endowments

$$x_A + x_B = \bar{x}_A + \bar{x}_B = 10$$

$$y_A + y_B = \bar{y}_A + \bar{y}_B = 20$$

(c) **Positive prices**

Prices must satisfy

$$p_x > 0 \qquad p_y > 0$$

We can now quickly verify that the equilibrium found before

$$\frac{p_x}{p_y} = 2 \qquad (x_A, y_A) = (8, 8) \qquad (x_B, y_B) = (2, 12)$$

satisfies all these conditions

First, prices are positive if we normalize, for example, to

$$p_y = 1 \qquad p_x = 2$$

Second, the allocation is feasible since

$$x_A + x_B = 8 + 2 = 10$$

$$y_A + y_B = 8 + 12 = 20$$

Third, each consumer exhausts their budget

For consumer A,

$$p_x x_A + p_y y_A = 2 \cdot 8 + 1 \cdot 8 = 24$$

$$p_x \bar{x}_A + p_y \bar{y}_A = 2 \cdot 8 + 1 \cdot 8 = 24$$

For consumer B,

$$p_x x_B + p_y y_B = 2 \cdot 2 + 1 \cdot 12 = 16$$

$$p_x \bar{x}_B + p_y \bar{y}_B = 2 \cdot 2 + 1 \cdot 12 = 16$$

Hence, the allocation before together with the relative price $\frac{p_x}{p_y} = 2$ satisfies all the conditions of a Walrasian equilibrium

4. To find the contract curve, we characterize the Pareto efficient allocations

The total endowment is

$$x_A + x_B = 10 \qquad y_A + y_B = 20$$

At an interior Pareto efficient allocation, the marginal rates of substitution must be equal

For consumer A,

$$MRS_A = \frac{MU_{y_A}}{MU_{x_A}} = \frac{4/y_A}{1} = \frac{4}{y_A}$$

For consumer B,

$$MRS_B = \frac{MU_{y_B}}{MU_{x_B}} = \frac{6/y_B}{1} = \frac{6}{y_B}$$

Thus efficiency requires

$$\frac{4}{y_A} = \frac{6}{y_B}$$

Using feasibility in good y,

$$y_B = 20 - y_A$$

so

$$\frac{4}{y_A} = \frac{6}{20 - y_A}$$

$$4(20 - y_A) = 6y_A$$

$$80 - 4y_A = 6y_A$$

$$10y_A = 80$$

$$y_A = 8 \qquad y_B = 12$$

Notice that utility is linear in good x for both consumers, with the same marginal utility equal to 1. Therefore, once the efficient allocation of good y is fixed at $(8, 12)$, any redistribution of good x between the two consumers is Pareto efficient, as long as total x remains equal to 10

Hence the contract curve is

$$\mathcal{C} = \left\{(x_A, y_A) \in \mathbb{R}^2_+ : y_A = 8,\ 0 \le x_A \le 10\right\}$$

Equivalently, in terms of both consumers' bundles,

$$(x_A, 8), (10 - x_A, 12) \quad \text{for } 0 \le x_A \le 10$$

Now we find the utility possibility frontier

Along the contract curve,

$$U_A = x_A + 4\ln(8)$$

$$U_B = (10 - x_A) + 6\ln(12)$$

Eliminating x_A from the first equation,

$$x_A = U_A - 4\ln(8)$$

Substituting into U_B,

$$U_B = 10 - (U_A - 4\ln(8)) + 6\ln(12)$$

$$U_B = 10 + 4\ln(8) + 6\ln(12) - U_A$$

This is the utility possibility frontier

Its feasible range is determined by $0 \le x_A \le 10$, which implies

$$4\ln(8) \le U_A \le 10 + 4\ln(8)$$

and correspondingly

$$6\ln(12) \le U_B \le 10 + 6\ln(12)$$

Thus the UPF is the line segment connecting the points

$$(4\ln(8),\ 10 + 6\ln(12))$$

$$(10 + 4\ln(8),\ 6\ln(12))$$

Therefore, the contract curve is the set of allocations with $y_A = 8$ and $y_B = 12$, while the utility possibility frontier is the straight line $U_B = 10 + 4\ln(8) + 6\ln(12) - U_A$

5. From the first part, the equilibrium allocation is

$$(x_A, y_A) = (8, 8) \qquad (x_B, y_B) = (2, 12)$$

The contract curve is

$$\mathcal{C} = \{(x_A, y_A), (x_B, y_B) : y_A = 8,\ y_B = 12,\ x_A + x_B = 10\}$$

The allocation found satisfies

$$y_A = 8 \qquad y_B = 12 \qquad x_A + x_B = 8 + 2 = 10$$

Therefore, the equilibrium allocation belongs to the contract curve

Since every allocation on the contract curve is Pareto efficient, it follows that the equilibrium allocation found in the first part is Pareto efficient

We can also verify this using the utility possibility frontier

$$U_B = 10 + 4\ln(8) + 6\ln(12) - U_A$$

At the equilibrium allocation,

$$U_A = 8 + 4\ln(8)$$

$$U_B = 2 + 6\ln(12)$$

Substituting U_A into the frontier,

$$U_B = 10 + 4\ln(8) + 6\ln(12) - (8 + 4\ln(8))$$

$$U_B = 2 + 6\ln(12)$$

which is exactly the utility of consumer B at the equilibrium allocation

Thus, the equilibrium utility pair lies on the utility possibility frontier

Hence, the equilibrium found in the first part is efficient because the allocation lies on the contract curve, and equivalently, the associated utility pair lies on the utility possibility frontier